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1998 J. Phys. A: Math. Gen. 31 3019

(http://iopscience.iop.org/0305-4470/31/13/010)

J. Phys. A: Math. Gen. **31** (1998) 3019–3039. Printed in the UK PII: S0305-4470(98)81926-4

An atom in a superstrong laser field

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Received 18 February 1997, in final form 2 December 1997

Abstract. The consistent analytical theory of multiphoton ionization in a superstrong laser field is developed. The corresponding initial problem for the Schrödinger equation is reduced to the initial problem for the pair of integro-differential equations. The appropriate asymptotic technique is derived and the leading asymptotic term of the solution is obtained. It is shown, that the atomic stabilization is present under fairly general conditions. The possibility of population revivals is also shown.

1. Introduction

Such phenomena as photodetachment of negatively charged ions, photoionization of atoms and photodissociation of molecules can be considered with the aid of the non-stationary Schrödinger equation. Because of this we can discuss these physical problems using a similar approach and by using general terminology. In the case of an (ultra)strong laser field perturbation theory cannot be used and we have to devise new methods to find solutions of the Schrödinger equation. Some attempts have been made to derive a theory of atomic ionization under a strong laser field by using quasiclassical and semiclassical considerations or by reducing the analysis to numerical experiments [1–8]. Up until now, however, no consistent analytical theory for describing the ionization of atoms in the superstrong laser field has been suggested. This paper is an attempt to build such a theory by exploiting the methods of [9–11]. A short presentation of this approach was published in [12].

Roughly speaking, the dynamics of the above-mentioned physical systems under a laser field are determined by values of the following parameters: the Rabi parameter R, the distinctive frequency of external radiation ω and some parameter D, which effectively describes the structure of the static part of the atomic potential. Each parameter here needs rigorous definition. Omitting the cumbersome details connected with such definitions (some of them will be discussed later), we can say that the relations between these parameters determine the regime of ionization and, respectively, the methods of theoretical investigations of these processes. If $\omega \ll R, D$, we have adiabatic ionization and need an appropriate approach. If $R \ll \omega, D$, we get a situation with a weak external laser field and can use the corresponding version of perturbation theory. If $R, D \ll \omega$, we have a high-frequency situation and the Kramers–Henneberger approach is a suitable framework. If $\omega, D \ll R$, the external laser field can be called a superstrong field. Precisely this situation will be considered later. In this case after scaling we obtain a pair of integro-differential equations with a large parameter. In order to construct the solution of our physical problem we need to use an appropriate asymptotic procedure. One aim of this paper is to derive

0305-4470/98/133019+21$19.50

such an asymptotic approach. We will discuss this problem on physical standards of rigor using rather formal methods.

We examine here the most simple physical situation. Namely, we consider the one-dimensional atom and suppose that only one discrete level of the static part of the atomic potential is populated initially. Moreover, we assume, that a one-electron model is valid. The other assumptions (about external field, the structure of atomic potential and so on) are fairly general. We will discuss later the possible generalizations of our formalism.

The organization of the paper is as follows. In the next section we introduce the main equations. The basis of our approach is the reduction of the Schrödinger equation to the pair of integro-differential equations. After scaling, these equations produce a large parameter proportional to the amplitude of the laser field. In order to calculate the leading asymptotic term of the solution we derive in section 3 the truncated equation. Sections 4 and 5 are devoted to solving this equation. Here we obtain our main result (relation (51)) which describes the solution in explicit analytical terms. In section 6 we will consider the physical consequences of this result. In section 7 we discuss our findings and pose some open problems. Some cumbersome mathematical details are displayed in the appendices.

2. Description of the basic formalism

We start from the non-stationary Schrödinger equation

$$\mathrm{i}\frac{\partial\Psi}{\partial t} = \{H_0 + A(t)x\}\Psi. \tag{1}$$

Here H_0 is a static part of the atomic potential (which can contain the contribution of the static external fields), $A(t)x$ is a contribution of the external laser field (in the dipole approximation). Let $|0\rangle$ be the initially populated level, and $|E\rangle$ be the other energy levels of H_0 (both discrete states and states in the continuous spectrum). Let the atom's wavefunction $\Psi(t)$ be a linear combination of these states

$$\Psi(t) = a(t)|0\rangle + \int b(E,t)|E\rangle\,\mathrm{d}E. \tag{2}$$

In this and in what follows the integral with respect to E also incorporates (when necessary) a sum over discrete states (we drop the limits of the integration). By a shift in the energy scale we can put the energy of the zeroth level equal to zero. E is the energy of the state $|E\rangle$. Let $A(t) = \rho q(t)$, and

$$q(t) = \int_{\omega_1}^{\omega_2} \mu(\hat{\omega})\cos[\hat{\omega}t + \varphi(\hat{\omega})]\,\mathrm{d}\hat{\omega}$$

in which the optical frequencies ω_1 and ω_2 are of the same order, and ρ and $q(t)$ are chosen so that $\max|q(t)| = 1$. (To be more definite one can assume, for instance, that $q(t) = \cos\omega t$.) The external laser field in our model can be modulated in an arbitrary manner. This means that it may also contain several modulated optical harmonics with frequencies of the same order. Note, that examination of multiphoton processes is beyond the scope of the rotating wave approximation.

We next choose the time scale so that $\omega = 1$, where generally $\omega = (\omega_1 + \omega_2)/2$. In terms of the amplitudes $a(t)$ and $b(E,t)$ equation (1) has the form

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t)\left[G_0 + \int g(E)b(E,t)\,\mathrm{d}E\right] \tag{3}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -\mathrm{i}Eb(E,t) + \mathrm{i}\rho q(t)\left[g(E)a(t) + \int U(E,E_1)b(E_1,t)\,\mathrm{d}E_1\right]. \tag{4}$$

Here G_0, $g(E)$ and $U(E, E_1)$ are the respective matrix elements of the atom's dipole momentum operator. For instance

$$G_0 = \langle 0|x|0\rangle.$$

Equations (3) and (4) constitute an infinite system of integro-differential equations: the spectral parameter E assumes an infinite set of values.

We discuss in this paper the one-dimensional situation, when the physical sources of the external field can be the neighbouring surface or neighbouring atom(s) etc. Evidently, in these situations $G_0 \neq 0$. But this value is not an important parameter of our formalism and does not play a serious role in the following. One more source of the static external field in the one-dimensional case is the static electric field, but if it is present the static part of the potential H_0 has no discrete level. It must be remembered that existence of at least one discrete level (one of them is populated initially and was denoted as $|0\rangle$) is a necessary condition of our statement of the problem. If the static electric field is present our statement of the initial problem itself has apparent obstacles.

We are interested here in the solutions of (3) and (4) that satisfy the initial conditions

$$a(0) = 1 \tag{5}$$

$$b(E, 0) = 0. \tag{6}$$

More complex initial conditions we will discuss elsewhere.

The parameter ρ in the above equations has yet to be specified: we can multiply ρ and divide $g(E)$ simultaneously by an arbitrary factor. We fix ρ by assuming that

$$\int g^2(E)\,\mathrm{d}E = 1. \tag{7}$$

We call the corresponding value of the dimensionless parameter ρ the effective Rabi parameter of our problem, and in order to obtain the dimensional value of the Rabi parameter R it must be multiplied by ω.

Information about the static part of the atomic potential in our model is contained in the functions $g(E)$ and $U(E, E_1)$. We will suppose, that these functions satisfy the following conditions

$$g(E) \sim \theta E^{-\alpha-1} \qquad E \to \infty \qquad U(E, E_1) \sim \nu|E - E_1|^{-\alpha-1}$$
$$|E - E| \to \infty \qquad 0 < \alpha < 1/2. \tag{8}$$

Let us formulate our asymptotic suggestions as two conditions.

Condition 1. We assume, that

$$\rho \gg 1. \tag{9}$$

Condition 2. Functions $g(E)$ and $U(E, E_1)$ are of order unity (more precisely, this means that parameters θ, ν and all moments of these functions that are not infinite are of order unity).

These conditions describe our range of physical parameters in the dimensionless form.

Note, that when we study photoionization under a weak external field the ionization potential I_P (and its relation with ω) plays an important role. However, under a superstrong laser field it is not enough to describe H_0 using one parameter only. We need here more detailed information about H_0, namely, we have to use the full (infinite) set of moments of the functions $g(E)$ and $U(E, E_1)$. Because of this we use D, an 'effective width of the spectrum of H_0', as a unique parameter classifying H_0. (We mean this value in the initial dimensional form; we have scaled our problem by the condition $\omega = 1$.) From this point of view, D is a more general characteristic of H_0 than I_P. Using this notation, we can rewrite

condition 1 as $R \gg \omega$, and condition 2 as $D/\omega = \mathrm{O}(1)$ (in initial dimensional notation), keeping in mind, that the last relations are the brief form of conditions 1 and 2 only.

Taking into account the initial condition (6), we put

$$b(E,t) = \mathrm{i}\rho \int_0^t a(x)q(x)\Xi(E,x,t)\,\mathrm{d}x. \tag{10}$$

Substituting (10) into (4), we obtain an equation for the transfer function $\Xi(E,x,t)$ and an initial condition

$$\frac{\partial \Xi(E,x,t)}{\partial t} = -\mathrm{i}E\,\Xi(E,x,t) + \mathrm{i}\rho q(t) \int U(E,E_1)\Xi(E_1,x,t)\,\mathrm{d}E_1 \tag{11}$$

$$\Xi(E,t,t) = g(E). \tag{12}$$

Using (10), equation (3) can be written as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t)\left[G_0 + \mathrm{i}\rho \int_0^t a(x)q(x)\Phi(x,t)\,\mathrm{d}x\right] \tag{13}$$

in which

$$\Phi(x,t) = \int g(E)\Xi(E,x,t)\,\mathrm{d}E. \tag{14}$$

We have therefore reduced the problem of multiphoton ionization in a superstrong laser field to finding the solution of the initial-value conditions (5) and (12) for the pair of integro-differential equations (11) and (13). These equations contain a large parameter ρ (a dimensionless Rabi parameter) which suggests using an asymptotic procedure to find the solution. The key component of this procedure is to perform an asymptotic expansion of the kernel of the integral operator on the right-hand side of (13), integrating by parts. It is sufficient for us to construct the two leading terms in an approximation of this integral operator. These terms (which are integral operators) have a simple structure, which makes it possible to find the necessary asymptotic approximation of the solution of our initial-value problem. Following the terminology of [9], we say that the function is fast (slow) if differentiation raises (leaves unchanged) its asymptotic order. This terminology is convenient in grading the different terms in equations.

The method for constructing an asymptotic expansion of an integral of the product of a fast function and a slow one is well known [13]. One needs only to integrate by parts, shifting differentiation from the fast function to the slow one. This operation produces terms with sequentially lowering order. Properly modified, this method was used in [14] to construct the full asymptotic solution of the integro-differential equation with a slow kernel without singularities. More complex situations were discussed in [9] and [10], where the dynamics of the level-continuum and level-band under a strong modulated laser field were examined. Namely, in [9] it was considered as an integro-differential equation with turning point singularity (which corresponds to the presence of the infinite continuum), and in [10] it was discussed as an integro-differential equation with a singularity-like turning point (which arises when the external field has a deep modulation). The situation discussed in this paper presents a new kind of singularity: the kernel of the integral on the right-hand side of (13) is a fast function. Indeed, as follows from (13), (14) and (11), the functions $a(x)$ and $\Phi(x,t)$ are fast functions and integration by parts produces terms of the same asymptotic order as the initial ones. Because of this, the methods used in [9] and [10] and [14] are inapplicable here and we have to derive a new technique.

The main technical idea of our approach consists in the following (it was derived in [11] for the more simple model). Integrating by parts (shifting the differentiation from

$a(x)$ to $\Phi(x,t)$) an infinite number of times leads to an infinite series of terms of the same asymptotic order. If we collect all terms of the leading asymptotic order, we derive the integro-differential operator with a simple structure. Collecting terms of the next asymptotic order, which appear as a result of an infinite iteration of the integration by parts procedure, we obtain a correction for the right-hand side, which is another integral operator with a kernel that can be calculated explicitly. This procedure makes it possible to determine the leading terms in the asymptotic expansion of the solution of the initial-value problem equations (5), (11), (12) and (13). Here is some analogy with the ordinary WKB-expansion of the solution of an ordinary differential equation with a large parameter. The asymptotic behaviour of this solution has the following structure

$$u(t) = \exp\{\mathrm{i}\rho\mu_1(t) + \mu_2(t) + \mu_3(t)/\rho + \cdots\}.$$

The right-hand side of equation (1) contains a large parameter and from the 'naive' point of view it looks like an ordinary differential equation with a large parameter. Our goal is to find the analogues of $\mu_1(t)$ and $\mu_2(t)$, which together determine what we call the leading term in the asymptotic expansion of the solution, a solution to within a small correction term. Our first step here is the derivation of the truncated equation. We will derive it with a formal level of rigor.

3. Derivation of the truncated equation

In integrating by parts on the right-hand side of (13) we will need to calculate quantities of the form $\partial^k\Phi(x,t)/\partial x^k$, $k = 1, 2, 3\ldots$. Let us rewrite equations (11) and (12) in a more convenient form. Clearly, these equations are equivalent to the following equation

$$\Xi(E,x,t) = g(E)\exp[-\mathrm{i}E(t-x)] + \mathrm{i}\rho\int_x^t q(s)\exp[-\mathrm{i}E(t-s)]\,\mathrm{d}s\int U(E,E_1)\Xi(E_1,x,s)\,\mathrm{d}E_1. \tag{15}$$

When combined with (14), this equation makes it possible for us to effectively calculate the derivatives of $\Phi(x,t)$. To illustrate our procedure, we perform a double integration of equation (13) by parts. For brevity we introduce a new operator D as follows

$$Dm(t) = [\mathrm{i}\rho q(t)]^{-1}\frac{\mathrm{d}m(t)}{\mathrm{d}t}. \tag{16}$$

If we introduce a new variable ('fast' variable)

$$p(t) = \mathrm{i}\rho\int_0^t q(s)\,\mathrm{d}s \tag{17}$$

then

$$v(t) = [Dm](p) = \frac{\mathrm{d}m}{\mathrm{d}p}.$$

We can assume here that

$$[D^{-1}v](x) = \int_0^x v(p)\,\mathrm{d}p. \tag{18}$$

It follows from equation (16) that

$$\mathrm{i}\rho q(t)v(t) = \frac{\mathrm{d}(D^{-1}v)}{\mathrm{d}t}. \tag{19}$$

Thus, according to (19), equation (13) can be written as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t)\left[G_0 + \int_0^t \left\{\frac{\mathrm{d}}{\mathrm{d}x}[D^{-1}a]\right\}(x)\Phi(x,t)\,\mathrm{d}x\right]$$

and integrating by parts this gives

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t)\left\{G_0 + [D^{-1}a](t)\Phi(t,t) - \int_0^t [D^{-1}a](x)\frac{\partial\Phi(x,t)}{\partial x}\,\mathrm{d}x\right\}. \tag{20}$$

The second integrated term vanishes due to our choice of the low limit of integration in (18). Combining (14) and (15) we find

$$\Phi(t,t) = \int g^2(E)\mathrm{d}E \equiv G_1. \tag{21}$$

As follows from the right-hand side of (20), we have to determine the value of $\partial\Phi(x,t)/\partial x$. Calculating $\partial\Xi(E,x,t)/\partial x$ by equation (15), we get

$$\frac{\partial\Xi(E,x,t)}{\partial x} = \phi_1(E,x,t) + \mathrm{i}\rho q(x)\Psi_1(E,x,t) \tag{22}$$

where

$$\phi_1(E,x,t) = \mathrm{i}Eg(E)\exp[-\mathrm{i}E(t-x)]$$
$$\Psi_1(E,x,t) = -\int U(E,E_1)g(E_1)\,\mathrm{d}E_1.$$

Using equations (22) and (14), we can now calculate the integrand on the right-hand side of (20). Integrating the term containing $\Psi_1(E,x,t)$ by parts in the resulting integral, we arrive at

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t)\left\{G_0 + [D^{-1}a](t)G_1 + [D^{-2}a](t)G_2 - \mathrm{i}\int_0^t [D^{-1}a](x)\psi_1(x,t)\,\mathrm{d}x + \int_0^t [D^{-2}a](x)\frac{\partial\hat{\Psi}_1(x,t)}{\partial x}\,\mathrm{d}x\right\}$$

$$\hat{\Psi}_1(x,t) = \int g(E)\Psi_1(E,x,t)\,\mathrm{d}E$$

$$\psi_1(x,t) = \int Eg^2(E)\exp[-\mathrm{i}E(t-x)]\,\mathrm{d}E$$

$$G_2 = \int\int \mathrm{d}E\,\mathrm{d}E_1 g(E)U(E,E_1)g(E_1).$$

We continue this procedure bearing in mind the following:

(a) integration of a fast function lowers its asymptotic order;

(b) if $a(x)$ is a fast function and $q(x)$ and $\psi(x)$ are slow functions, then

$$\int_0^t \mathrm{d}x\,a(x)\int_x^t \mathrm{d}s\,q(s)\psi(s) = \int_0^t \mathrm{d}x\,a(x)q(x)\int_x^t \mathrm{d}s\,\psi(s)[1+o(1)].$$

As a result, collecting the terms of the asymptotic expansions of the integral operator on the right-hand side of equation (13), which have two leading orders, we arrive at the truncated equation

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t)\left\{\sum_{k=0}^{\infty}[D^{-k}a](t)G_k - \mathrm{i}\int_0^t \mathrm{d}x\sum_{k=1}^{\infty}[D^{-k}a](x)\psi_k(x,t)\right\} \tag{23}$$

where for $k = 1, 2, \ldots$

$$G_k = \int \cdots \int \left(\prod_{m=1}^{k} \mathrm{d}E_m \right) g(E_1) \left(\prod_{n=1}^{k-1} U(E_n, E_{n+1}) \right) g(E_k)$$

$$\psi_k(x,t) = \int \cdots \int \left(\prod_{m=1}^{k} \mathrm{d}E_m \right) g(E_1) \exp[-\mathrm{i}E_1(t-x)] \left(\prod_{n=1}^{k-1} U(E_n, E_{n+1}) \right) g(E_k) \sum_{p=1}^{k} E_p.$$

Note, that the above two terms are indeed two successive terms in the asymptotic expansion of the integral term on the right-hand side of (13): the second term, in comparison to the first, contains one more integration with respect to x, which lowers the asymptotic order.

To find the leading asymptotic term of the solution of the initial-value problem (5), (11), (12) and (13) it is sufficient to construct a solution of the truncated equation (23) that satisfies condition (5).

4. Solving the truncated equation: first step

We start with the leading term of the asymptotic expansion (23), i.e. the equation

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \mathrm{i}\rho q(t) \sum_{k=0}^{\infty} [D^{-k}a](t) G_k. \tag{24}$$

Let us now construct a solution of this equation that satisfies condition (5). We shift to the variable p according to (26) and integrate this equation once allowing for the initial value (5). Then, we arrive at a Volterra integral equation of the second kind

$$a(p) - \sum_{k=0}^{\infty} [D^{-k-1}a](p) G_k = 1 \tag{25}$$

with D^{-1} defined in (18). In order to solve this equation we use the Laplace transformation and find $\hat{a}(r)$, the Laplace transform of the function $a(p)$ (see [15])

$$\hat{a}(r) = [r(1 - K(r))]^{-1} \qquad K(r) = \sum_{k=0}^{\infty} G_k r^{-1-k}. \tag{26}$$

We can rewrite this function in a more explicit form. Let λ_n and $\mu_n(E)$ be the eigenvalues and normalized eigenfunctions of the operator $\mathfrak{U}$, which one can call the submatrix of the atom's dipole momentum operator

$$(\mathfrak{U}\mu)(E) = \int U(E, E_1)\mu(E_1)\,\mathrm{d}E_1.$$

Let us suppose at first, for brevity, that $\mathfrak{U}$ has a purely discrete spectrum; its spectrum, containing a continuous part, will be described in full details later. The function $U(E, E_1)$ is symmetric, so that $\lambda_n = \mathrm{Re}\,\lambda_n$, and the eigenfunctions form a orthonormal base in the respective function space. We expand the function $g(E)$ in these base functions

$$g(E) = \sum_n g_n \mu_n(E) \tag{27}$$

then

$$G_k = \sum_n g_n^2 \lambda_n^{k-1} \qquad k = 1, 2, \ldots$$

and

$$K(r) = G_0 r^{-1} + \sum_n g_n^2 [r(r - \lambda_n)]^{-1}. \tag{28}$$

As a result we obtain the following expression

$$\hat{a}(r) = \left[r - G_0 - \sum_n g_n^2 (r - \lambda_n)^{-1} \right]^{-1}. \tag{29}$$

We must obtain this formula for the case of the real atomic potential (i.e. for the real function $U(E, E_1)$). In order to do this we need information about the spectrum of the operator $\mathfrak{U}$. It is more convenient to describe this operator in the spatial representation, and to assume that the atom is one-dimensional.

Let $\psi_0(x) = |0\rangle$ be the normalized wavefunction of the initially populated level. Let S be the projection operator on the one-dimensional space associated with this function. Then, according to our definition

$$\mathfrak{U}\psi(x) = (I - S)x(I - S)\psi(x)$$

where x is the operator of multiplication by the variable x, and I is the identity operator. The eigenfunctions of $\mathfrak{U}$ satisfy the following equations

$$x\phi(x, \mu) = \mu\phi(x, \mu) + \kappa(\mu)\psi_0(x) \tag{30}$$

$$\int \phi(\mu, x)\psi_0^*(x)\,\mathrm{d}x = 0. \tag{31}$$

Equation (30) immediately yields

$$\phi(\mu, x) = \delta(x - \mu) + \kappa(\mu)\psi_0(x)(x - \mu)^{-1} \tag{32}$$

where $-\infty < \mu < \infty$ and $\psi_0(x)(x-\mu)^{-1}$ we interpret as a distribution $P\{\psi_0(x)(x-\mu)^{-1}\}$, where P is the Cauchy principal value. Substituting (32) into (31) gives

$$\kappa(\mu) = -\psi_0^*(\mu)\left[P \int |\psi_0(x)|^2 (x - \mu)^{-1}\,\mathrm{d}x \right]^{-1}. \tag{33}$$

Relations (32) and (33) describe the continuous spectrum of the operator $\mathfrak{U}$. The corresponding eigenfunction must be normalized. This normalization can be made with the help of the modified Poincare–Bertrand calculation (see appendix A). It is shown there, that

$$\int \mathrm{d}x\, \phi(\mu, x)\phi^*(\lambda, x) = [1 + \pi^2 |\kappa(\mu)\psi_0(\mu)|^2]\delta(\lambda - \mu). \tag{34}$$

Letting

$$\alpha(\mu) = \mathrm{sign}\left\{ P \int |\psi_0(x)|^2 (x - \mu)^{-1}\,\mathrm{d}x \right\}$$

then

$$\Phi(\mu, x) = \phi(\mu, x)T(\mu)$$
$$T(\mu) = [1 + \mathrm{i}\pi\alpha(\mu)|\kappa(\mu)\psi_0(\mu)|]^{-1}$$

are eigenfunctions of the continuous spectrum, $-\infty < \mu < \infty$.

Besides the continuous spectrum, operator $\mathfrak{U}$ can have discrete eigenvalues. If there are such μ_k, $k = 1, 2, \ldots, N$, so that the following relations are valid simultaneously

$$\psi_0(\mu_k) = 0 \tag{35}$$

$$P \int |\psi_0(x)|^2 (x - \mu_k)^{-1}\,\mathrm{d}x = 0 \tag{36}$$

then functions

$$\phi_k(x) = \psi_0(x)(x - \mu_k)^{-1} \tag{37}$$

are eigenfunctions of the discrete spectrum of the operator $\mathfrak{U}$. If $\Phi_k(x)$, $k = 1, 2, \ldots, N$, are normalized in the $L^2(R)$ versions of these functions, one more eigenfunction is $\psi_0(x)$, which corresponds to the discrete eigenvalue $\mu = 0$. Then

$$\{\Phi(\mu, x), -\infty < \mu < \infty, \Phi_k(x), k = 1, 2, \ldots, N, \psi_0(x)\}$$

is the full set of the eigenfunctions of $\mathfrak{U}$.

Equation (36) can have roots. If, for instance, $\psi_0(x)$ is symmetric with respect to the substitution $x \to -x$, then $\mu = 0$ is a solution of (36). Moreover, if the static potential H_0 has Coulomb singularity at $x = 0$, then $\psi_0(0) = 0$ and $\mu_1 = 0$ is the discrete eigenvalue of $\mathfrak{U}$.

So, the eigenfunctions of operator $\mathfrak{U}$ are distributions and we have to use the corresponding version of the spectral theorem (see [16–18]). According to this theorem, one can represent 'any' function as a linear combination of these eigenfunctions (Parceval's formula). Our aim is to derive the analogues of relations (27) and (28). We can use the spatial representation directly. It follows from our definitions that we have to expand the function $(I - S)x\psi_0(x)$. Thus, we obtain the coefficients of this expansion on continuous eigenfunctions and discrete eigenfunctions respectively

$$c(\mu) = \int x\psi_0(x)\Phi^*(\mu, x)\,\mathrm{d}x = \kappa^*(\mu)T^*(\mu)$$

$$c_k = \int x\psi_0(x)\Phi_k^*(x)\,\mathrm{d}x = \|\phi_k(x)\|^{-1}.$$

We can now write the analogue of (28) in the form

$$K(r) = r^{-1}\left[G_0 + P\int \frac{\mathrm{d}\mu|\kappa(\mu)T(\mu)|^2}{r-\mu} + \sum_{k=1}^{N}\frac{|c_k|^2}{r-\mu_k}\right]. \tag{38}$$

The analogue of relation (29) is

$$\hat{a}(r) = \left[r - G_0 - P\int \frac{\mathrm{d}\mu|\kappa(\mu)T(\mu)|^2}{r-\mu} - \sum_{k=1}^{N}\frac{|c_k|^2}{r-\mu_k}\right]. \tag{39}$$

In order to invert the Laplace transform and find the function $a(p)$, we will use a general approach [15]. We have to choose as the integration contour on the complex plane r some straight line parallel to the imaginary axis, which is situated to the right of the support of the integrand. One can find the details of this calculation in appendix B. Here we present the result only.

Let $\zeta_m, m = 1, 2, \ldots, M$, be the solutions of the transcendent equation

$$r = G_0 + P\int \frac{\mathrm{d}\mu|\kappa(\mu)T(\mu)|^2}{r-\mu} + \sum_{k=1}^{N}\frac{|c_k|^2}{r-\mu_k}. \tag{40}$$

We will call these solutions pseudopoles. Let

$$B_m = \lim_{r\to\zeta_m}\left\{(r-\zeta_m)\left[r - G_0 - P\int \frac{\mathrm{d}\mu|\kappa(\mu)T(\mu)|^2}{r-\mu} - \sum_{k=1}^{N}\frac{|c_k|^2}{r-\mu_k}\right]^{-1}\right\} \tag{41}$$

these values are the corresponding pseudoresidues. Then

$$a(t) = \sum_{m=1}^{M} B_m \exp\left[\mathrm{i}\rho\zeta_m\int_0^t q(s)\,\mathrm{d}s\right] + \int_{-\infty}^{\infty}\mathrm{d}r\exp\left[\mathrm{i}r\rho\int_0^t q(s)\,\mathrm{d}s\right]\Lambda(r) \tag{42}$$

in which

$$\Lambda(r) = \pi|\kappa(r)T(r)|^2\left\{\left[r - G_0 - P\int\frac{\mathrm{d}\mu|\kappa(\mu)T(\mu)|^2}{r-\mu} - \sum_{k=1}^{N}\frac{|c_k|^2}{r-\mu_k}\right]^2 + \pi^2|\kappa(r)T(r)|^2\right\}^{-1}.$$

It follows from the explicit form that

$$|\Lambda(r)| = O(|\kappa(r)|^2 r^{-2}) = O(|\psi_0(r)|^2) \qquad \text{at} \quad |r| \to \infty.$$

Because of this we can use relation (42) with infinite limits of integration.

Let us now discuss the number of pseudopoles M. They are solutions of equation (40). In order to estimate this quantity, we have to take into account the following considerations:

(i) the left-hand side of equation (40) is an unbounded function;

(ii) the singularities of the right-hand side of this equation are placed in the discrete eigenvalues of the operator $\mathfrak{U}$;

(iii) the integral part on the right-hand side of (40) is a bounded continuous function.

If $N = 0$, then the right-hand side of (40) is bounded, and one can see that at any rate one solution of this equation exists. Let $N \geqslant 1$ and note that $|c_k|^2 > 0$ for any k. We conclude, that in some vicinity of each μ_k at least one solution of ζ_m exists (see figure 1). Ultimately we get $M \geqslant N + 1$.

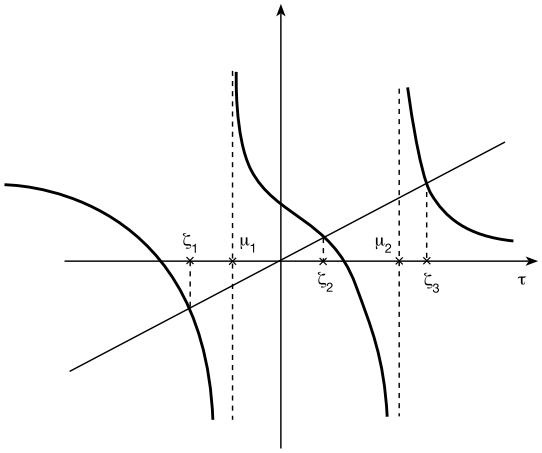

Figure 1.

5. Solving the truncated equation: second step

Thus, we have obtained the solution of equation (24) satisfying the initial condition (5). Here we will consider the full truncated equation (23). We will seek its solution in the form

$$a(t) = \sum_{m=1}^{M} B_m \exp[S(t,\zeta_m)] + \int_{-\infty}^{\infty}\mathrm{d}r\exp[S(t,r)]\Lambda(r) \tag{43}$$

in which

$$S(t,r) = \mathrm{i}r\rho\int_0^t q(s)\,\mathrm{d}s + \int_0^t m(s,r)\,\mathrm{d}s.$$

This relation is a natural generalization of (42). The initial condition (5) is fulfilled automatically. In order to find the function $a(t)$ we have to determine the function $m(s, r)$ and, first of all, to find an equation for it.

We can discuss any exponent in (43) separately assuming that the function $m(s, r)$ is a slow function. Substituting (43) into (23) gives

$$[\mathrm{i}\rho r q(t) + m(t, r)] \exp[S(t, r)] = \mathrm{i}\rho q(t) \left\{ \int_0^t \mathrm{i}\rho q(x) Z \left[\mathrm{i}\rho \int_x^t q(s)\,\mathrm{d}s \right] \exp[S(x, r)]\,\mathrm{d}x + \int_0^t C(x, t) \exp[S(x, r)]\,\mathrm{d}x \right\} \tag{44}$$

where $Z(p - h)$ ($C(x, t)$) is a kernel of the first (second) integral operator on the right-hand side of (23). In essence the integral operator Z (with the corresponding kernel) was discussed in the preceding section. We will use here the following property of this operator

$$Z \exp(rp) = r \exp(rp). \tag{45}$$

Let us discuss in more detail the first term on the right-hand side of (44)

$$Q = \mathrm{i}\rho q(t) \int_0^t \mathrm{i}\rho q(x) Z \left[\mathrm{i}\rho \int_x^t q(s)\,\mathrm{d}s \right] \exp \left[\mathrm{i}\rho r \int_0^t q(s)\,\mathrm{d}s \right] \exp \left[\int_0^t m(s, r)\,\mathrm{d}s \right] \mathrm{d}x.$$

We integrate it by parts transferring the differentiation from Z to $\exp[\int_0^t m(s, r)\,\mathrm{d}s]$ and using (45)

$$Q = \mathrm{i}\rho r q(t) \exp[S(t, r)] - \mathrm{i}\rho r q(t) \int_0^t m(x, r) \exp[S(x, r)]\,\mathrm{d}x.$$

It is enough for us to calculate the integral in the last relation to leading asymptotic order. Let us rewrite Q in the following form

$$\mathrm{i}\rho r q(t) \int_0^t \mathrm{d}x\, m(x, r) \exp[S(x, r)] = \mathrm{i}\rho r \int_0^t \mathrm{d}x\, m(x, r) q(x) \exp[S(x, r)] + \mathrm{i}\rho r \int_0^t \mathrm{d}x\, m(x, r)[q(t) - q(x)] \exp[S(x, r)].$$

The second integrand on the right-hand side of the last relation has a root when $x = t$. It follows from this fact that the corresponding integral has the lowest asymptotic order [13]. The first integral on the right-hand side we integrate by parts and obtain to leading asymptotic order

$$\mathrm{i}\rho r \int_0^t \mathrm{d}x\, m(x, r) q(x) \exp[S(x, r)] = m(t, r) \exp[S(t, r)].$$

Collecting the terms in equation (44), we get an equation for $m(t, r)$

$$2m(t, r) = \mathrm{i}\rho q(t) \exp[-S(t, r)] \int_0^t C(x, t) \exp[S(x, r)]\,\mathrm{d}x.$$

As before, we can calculate the integral on the right-hand side to leading asymptotic order, omitting in $S(x, r)$ the term containing the function $m(x, r)$ and in $C(x, t)$ the dependence on the slow variables. Applying these considerations we find

$$m(t, r) = \text{constant}(t) = -\mathrm{i}V(r)/2 \tag{46}$$

where the function $V(r)$ is defined as follows: for r with a large positive real part

$$V(r) = \sum_{k=1}^{\infty} \delta_k r^{-k-1}$$

$$\delta_1 = \int E g^2(E)\, \mathrm{d}E$$

$$\delta_k = \int \cdots \int \left(\prod_{s=1}^{k} \mathrm{d}E_s \right) g(E_1) \left(\prod_{m=1}^{k-1} U(E_m, E_{m+1}) \right) g(E_k) \sum_{l=1}^{k} E_l.$$

For other r we use the result of the corresponding analytical continuation. Here we suppose, that $\delta_k = O(1)$ (it follows from our main asymptotic assumptions).

We can obtain a more explicit form of the function $V(r)$. Indeed, let us consider the solution of the integral equation

$$rf(E, r, z) = g(E) + \int \mathrm{d}E_1\, U(E, E_1) \exp[-E_1 z] f(E_1, r, z). \tag{47}$$

Then for large positive real r

$$V(r) = \frac{1}{r} \frac{\partial F(r, z)}{\partial z}\bigg|_{z=0}$$

in which

$$F(r, z) = \int \mathrm{d}E\, g(E) \exp[-Ez] f(E, r, z).$$

One can prove this relation by solving equation (47) by iteration. More convenient is the symmetric version of this integral equation. Substituting

$$f(E, r, z) = \exp[Ez/2] h(E, r, z)$$

we obtain for the function $h(E, r, z)$ the integral equation

$$rh(E, r, z) = g(E) \exp[-Ez/2] + \int \mathrm{d}E_1\, U(E, E_1) \exp[-(E_1 + E)z/2] h(E_1, r, z). \tag{48}$$

Let μ, $\vartheta(\mu, E; z)$ be eigenvalues and corresponding eigenfunctions of the integral operator

$$[\mathfrak{U}(z)h](E) = \int \mathrm{d}E_1\, U(E, E_1) \exp[-(E_1 + E)z/2] h(E_1, z). \tag{49}$$

It follows from our definitions that $\sigma(z)$, the spectrum of $\mathfrak{U}(z)$, tends to the spectrum of $\mathfrak{U}$ when $z \to 0$. Using the corresponding spectral expansion, we get from equation (48)

$$h(E, r, z) = \int_{\sigma(z)} \frac{\vartheta(\mu, E; z)}{r - \mu} \hat{g}(\mu, z)\, \mathrm{d}\mu$$

where

$$\hat{g}(\mu, z) = \int g(E) \exp[-Ez/2] \vartheta(\mu, E; z)\, \mathrm{d}E.$$

As a result we obtain

$$V(r) = \left[\frac{1}{r} \frac{\partial}{\partial z} P \int_{\sigma(z)} \frac{\hat{g}^2(\mu, z)\, \mathrm{d}\mu}{r - \mu} \right]\bigg|_{z=0}. \tag{50}$$

Actually we can calculate the function $V(r)$, using the information about operators $\mathfrak{U}$ and H_0 only. Indeed, in order to determine the right-hand side of relation (50) it is enough to consider instead $\mathfrak{U}(z)$ the more simple operator

$$[\hat{\mathfrak{U}}(z)h](E) = \int \mathrm{d}E_1 U(E, E_1)[1 - (E_1 + E)z/2]h(E_1, z).$$

We can find its spectral characteristics using a corresponding version of perturbation theory, and rewrite (50) in more explicit terms, but here we omit these cumbersome details.

Summing, we find the final relation, which is the main result of this work

$$a(t) = \sum_{m=1}^{M} B_m \exp[S(t, \zeta_m)] + \int_{-\infty}^{\infty} \mathrm{d}r \ \exp[S(t, r)]\Lambda(r) \tag{51}$$

in which

$$S(t, r) = \mathrm{i}r\rho \int_0^t q(s)\,\mathrm{d}s - \mathrm{i}tV(r)/2$$

and the function $\Lambda(r)$ has been described previously. It can be shown that the function $V(r)$ is bounded except for points, which are discrete eigenvalues of P. However, at these points function $\Lambda(r)$ equals zero.

Relation (51) describes the dynamics of the amplitude $a(t)$ at $t = O(1)$. The amplitudes $b(E, t)$ and the distribution of the electrons ejected from the atom can be obtained from (10). Explicit results will be described elsewhere.

6. Physical results: atomic stabilization and revivals

Here we discuss the physical consequences of our findings using the well known asymptotic results [13] about the integrals from fast-oscillating functions.

First, relation (51) describes the general structure of the solution of our initial value problem. Namely, if t does not equal the root of the equation

$$\int_0^t q(s)\,\mathrm{d}s = 0 \tag{52}$$

then the second term on the right-hand side of (51) is asymptotically small, and (51) reads

$$a(t) = \sum_{m=1}^{M} B_m \exp\left[\mathrm{i}\rho\zeta_m \int_0^t q(s)\,\mathrm{d}s - \mathrm{i}tV(\zeta_m)/2\right] + o(1). \tag{53}$$

It follows from this result that almost everywhere the solution is a finite sum of some simple functions up to an asymptotically small correction.

Let us now calculate the population of the level $|0\rangle$ (which was populated initially). Using (51) we obtain

$$n(t) = |a(t)|^2 = \sum_{m,n=1}^{M} B_m B_n{}^* \exp\left[\mathrm{i}\rho(\zeta_m - \zeta_n) \int_0^t q(s)\,\mathrm{d}s - \mathrm{i}t[V(\zeta_m) - V(\zeta_n)]/2\right] + o(1).$$

Averaging with respect to the fast oscillations gives

$$\hat{n}(t) = \sum_{m=1}^{M} |B_m|^2 + o(1). \tag{54}$$

As was discussed earlier, $M \geqslant 1$. This means, that the sum on the right-hand side of (54) is positive. Finally, we get

$$\hat{n}(t) = \text{constant} = O(1) > 0 \qquad t = O(1). \tag{55}$$

As follows from (51) (and was assumed previously, see conditions 1 and 2), the leading temporal parameter in our problem is ρ^{-1} and it is natural to estimate the duration of the level population decay as a value of order ρ^{-1}, so it must be $\hat{n}(t) = o(1)$ at $t = O(1)$. However, relation (55) shows that it is not valid. In accordance with it, this duration is much more than 1 (the period of the optical oscillations in our dimensionless notation) and increases when the intensity of the external field increases (see later discussions). Because of this one can interpret (55) as a trapping of the level population, or as an atomic stabilization.

Let us consider another physical consequence of our results. Let t_s, $s = 1, 2, \ldots$, be roots of equation (52). In the ρ^{-1}-vicinity of each point the integral part of the right-hand side of (51) is not asymptotically small. This means that in these vicinities the function $a(t)$ gives a sharp variation. Specifically, the level population can have sharp peaks (or gaps) in these vicinities. The widths of these singularities are of the order ρ^{-1}. We can interpret such peaks as revivals. The possible presence of revivals in the atomic ionization was discussed in [7]. Note, that as was mentioned previously the atomic characteristics oscillate with frequencies of order ρ (see (53)). Because of this the possibility of increasing the level population is a rather subtle phenomenon.

Let us next consider the simplest case where $q(t) = \cos t$. Then, $\int_0^t q(s)\,\mathrm{d}s = \sin t$ and we get $t_k = 2\pi k$. This situation looks like a return of the oscillating wavepacket to its initial position after each cycle. But this point of view is, in our opinion, too simplified. Namely, the wavepacket oscillates with frequencies like ρ^{-1} (as follows from (51)). From the point of view of these oscillations the roots t_k are very 'seldom' and the wavepacket can 'forget' the initial information. However, in these moments of time the sharp increasings (or gaps) of the level population can be due to some interference phenomenon. Note that values of the corresponding peaks (A_k) depends on subtle characteristics of the 'pseudopoles' and, generally speaking, can be non-monotonic relations between A_k and k (such situations cannot be described in terms of 'returns').

Figure 2 reflects the qualitative behaviour of the level population according to our results.

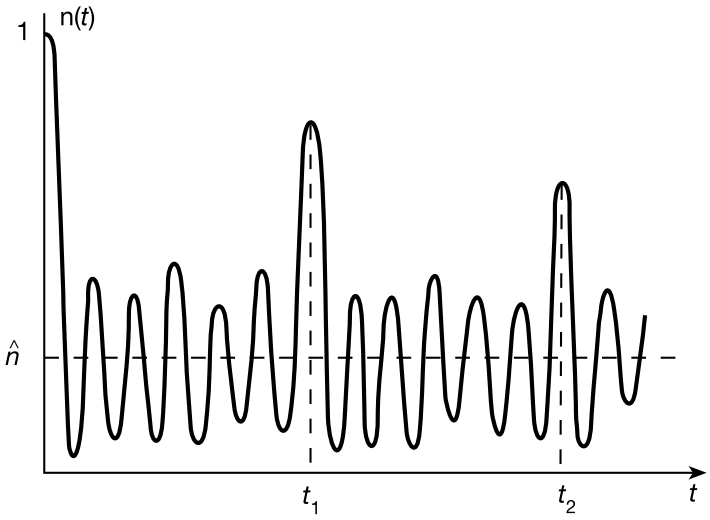

Figure 2.

7. Discussion

First, let us consider the possible role of subsequent terms of the asymptotic expansion of the solution. In the following asymptotic terms we face two singularities.

(i) The first one is connected with the asymptotic behaviour of the matrix elements of the atom's dipole momentum operator (8). If $0 < \alpha < 1/2$, then the next moments of functions $g(E)$ and $U(E, E_1)$ (such as $\int E^2 g^2(E)\,\mathrm{d}E$) are infinite. Such a singularity was considered in [9] in the simplest case. Namely, the quantum system level-continuum under a strong laser field was investigated with a disregard of the continuum–continuum transitions. It was shown that this singularity leads to the decay of the level population and the rate of this decay is generally speaking of order $\rho^{-2\alpha}$.

(ii) The second singularity is connected with zeros of the functional coefficient $q(t)$ (similar singularity is well known in the theory of ordinary differential equations with a large parameter). Such a singularity was discussed in the simplest variant (which corresponds to the physical situation: the quantum system level-band interacts with a deep modulated strong laser field) in [10]. It was shown there, that the presence of this singularity leads to the decay of the level population and that the rate of this decay is generally speaking of order ρ^{-1}. Thus an increase of the amplitude of the external field leads to a decrease of the decay rate of the level population.

Remark 1. The expression '... the rate of this decay is generally speaking...' already mentioned means the following. There are special initial conditions when the (average) level population is constant in the calculated asymptotic order (in both cases (i) and (ii)).

Remark 2. Note, that from the physical point of view the statements of problem of the atomic ionization in [9] and in [3] are very similar. In both papers the high-frequency situation was discussed, when the condition $\omega \gg R \gg D$ is valid. Using the Kramers–Henneberger framework and computational simulations it was shown in [3], that the 'average' atomic potential under a strong laser field gives a pair of discrete states, what can be interpreted as an atomic stabilization. In [9] the quantum system level-continuum under a strong modulated laser field was discussed and it was shown, that the solution is a linear combination of two generalized Rabi harmonics. But in [9], using a completely analytical approach, we calculate the rate of decay of the level population connected with the asymptotic behaviour of the matrix elements of the atom's dipole momentum operator.

In our situation there are three simultaneous singularities: fast-oscillating kernels of the integral part of the integro-differential equation (13) and the singularities (i) and (ii). Thus, the derivation of the next term of the asymptotic expansion of the solution is a harder problem. However, it follows from the qualitative considerations that in the present, more general case we can presuppose the analogous influence of the singularities (i) and (ii). In other words, taking into account the next terms in our problem we will obtain the decay rate of the level population of order $\max(\rho^{-1}, \rho^{-2\alpha})$. From the 'naive' point of view the atom under a superstrong laser field has to be fully ionized for the time period of order ρ^{-1}. However, it follows from our results, that the duration of ionization is of order $\rho^{\min(2\alpha,1)}$. This duration increases when the amplitude of the laser fields increased. Such phenomena are called atomic stabilizations. It is worthy to mention that the atomic ionization depends on many parameters of the atom and of the external field. In different regions of parameters atomic stabilization has different sources [8]. From the point of view developed here atomic stabilization (in the case when the laser field is superstrong) is the result of the necessary presence of the pseudopoles—solutions of equation (40).

We have solved our initial problem in explicit analytic terms connected with $\psi_0(x)$ and the full spectrum of the operators $\mathfrak{U}$ and H_0. This description is from some points of view the 'internal' one. How is this description connected with the 'external' description from the point of view of an outside observer? This connection is not simple for real physical systems. For instance, in order to calculate the effective Rabi parameter ρ introduced earlier we need information about the full spectrum of H_0 (or its effective estimate). Only after this can we verify the validity of our asymptotic assumptions (9) for the real physical systems. If we want to derive a computational procedure on the basis of this approach, we again need full information about the spectral description of the operator H_0. Obtaining this information can be difficult. That is why up to now our results are considered an analytical solution of the problem only. However, there are very broad sets of atom-like systems, which can be, in principle, examined in our terms, such as an atom in external static fields, negatively charged ions and so on. Some important characteristics of these systems can be chosen arbitrarily (for instance, an amplitude of the external static field). This means that for some real physical situations our results can give a sufficient description.

We have discussed here the one-dimensional atom whereas the real atom is a three-dimensional object. This assumption was used, when we considered the spectrum of the submatrix $\mathfrak{U}$. It as a unique point of our approach that this assumption is essential and means, that the description of the spectrum $\mathfrak{U}$ in the three-dimensional case is enough to expand our approach to the three-dimensional atom. It seems that the last problem can be solved in this way also.

There are other possible generalizations and applications of our results such as high-harmonic generation, multiple ionization (outside the one-electron model) and the relativistic approach (instead of the Schrödinger equation we can use the Dirac equation). Apparently, some modifications of our approach make it possible to rewrite our results for correspondent problems. One direction is connected with the choice of initial conditions and (or) the presence of another discrete level placed close to the level $|0\rangle$. As was discovered in [19] (see also [8]) in rather a different physical situation, field-induced interference stabilization can occur. So the question arises: is there an analogous phenomenon under a superstrong laser field? We will consider this in more detail elsewhere.

To summarize, a consistent analytical approach to the problem of the multiphoton ionization in a superstrong laser field has been suggested. The appropriate asymptotic procedure has been developed and the leading asymptotic term of the solution constructed. It has been shown that atomic stabilization exists under rather general assumptions about the external field and atomic structure. It has been noted that revivals of the level population can also be shown.

Acknowledgments

The author acknowledges fruitful discussions with V M Babich, M I Belishev, V S Buldyrev, N B Delone and M V Fedorov.

Appendix A

Our aim here is to derivate relation (34). Let us discuss the distribution $C(\mu, \nu)$, which acts on the test smooth function $r(\nu)$ in the following way

$$\int C(\mu, \nu) r(\nu)\, \mathrm{d}\nu = \int \mathrm{d}x\, \phi(\mu, x) \int \phi^*(\nu, x) r(\nu)\, \mathrm{d}\nu. \tag{56}$$

We will use the Sohotsky relation: for any test function $\zeta(x)$

$$P\int \frac{\zeta(x)\,\mathrm{d}x}{x-\mu} = \mathrm{i}\pi\theta\zeta(\mu) + \lim_{\varepsilon\to+0}\int \frac{\zeta(x)\,\mathrm{d}x}{x-\mu+\mathrm{i}\varepsilon\theta} \tag{57}$$

where $\theta = \pm 1$. The scheme is considered as follows. We will substitute the explicit expressions for the functions $\phi(\mu, x)$ and $\phi^*(\nu, x)$ into (56); then, using the Sohotsky relations, we will expand the ratio in the iterated integral into a sum of the ratios and obtain (34).

First, we discuss the distribution

$$S(x) = \int \phi^*(\nu, x) r(\nu)\,\mathrm{d}\nu.$$

This acts on any test function $u(x)$ in the following way

$$\langle S, u\rangle = \int \mathrm{d}\nu\, r(\nu)\int \mathrm{d}x\,\phi^*(\nu, x)u(x) = \int \mathrm{d}\nu\, r(\nu)u(\nu)$$
$$+\int \mathrm{d}\nu\, r(\nu)\kappa^*(\nu)\left[\mathrm{i}\pi\psi_0^*(\nu)u(\nu) + \lim_{\varepsilon\to+0}\int \frac{u(x)\psi_0^*(x)\,\mathrm{d}x}{x-\nu+\mathrm{i}\varepsilon}\right].$$

In the iterated integral we can change the order of integration (because $\varepsilon \neq 0$) and find

$$\langle S, u\rangle = \int \mathrm{d}\nu\, r(\nu)u(\nu) + \mathrm{i}\pi\int \mathrm{d}\nu\, r(\nu)\kappa^*(\nu)\psi_0^*(\nu)u(\nu)$$
$$+\lim_{\varepsilon\to+0}\int \mathrm{d}x\,\psi_0^*(x)u(x)\int \frac{\mathrm{d}\nu\, r(\nu)\kappa^*(\nu)}{x-\nu+\mathrm{i}\varepsilon}.$$

Using the Sohotsky formula once more (in the 'inverse' direction) gives

$$\langle S, u\rangle = \int \mathrm{d}\nu\, r(\nu)u(\nu) + \int \mathrm{d}x\,\psi_0^*(x)u(x)P\int \frac{\mathrm{d}\nu\, r(\nu)\kappa^*(\nu)}{x-\nu}.$$

We can rewrite this result in the form

$$S(x) = r(x) + \psi_0^*(x)P\int \frac{\mathrm{d}\nu\, r(\nu)\kappa^*(\nu)}{x-\nu}. \tag{58}$$

Let us now calculate the distribution $C(\mu, \nu)$ with the help of equation (58). We suppose for brevity that μ and ν are not roots of equation (36). From the definition of $C(\mu, \nu)$ we get

$$\int C(\mu, \nu)r(\nu)\,\mathrm{d}\nu = \int \mathrm{d}x\,\phi(\mu, x)\left[r(x) + \psi_0^*(x)P\int \frac{\mathrm{d}\nu\, r(\nu)\kappa^*(\nu)}{x-\nu}\right]$$
$$= r(\mu) + \kappa(\mu)P\int \frac{\mathrm{d}x\,\psi_0(x)r(x)}{x-\mu} + \psi_0^*(\mu)P\int \frac{\mathrm{d}\nu\, r(\nu)\kappa^*(\nu)}{\mu-\nu} + R(\mu) \tag{59}$$

in which

$$R(\mu) = \kappa(\mu)P\int \frac{\mathrm{d}x|\psi_0(x)|^2}{x-\mu}P\int \frac{r(\nu)\kappa^*(\nu)}{x-\nu}.$$

We next transform this expression. In order to split the singularities of the integrand we use the Sohotsky relation with $\theta = 1$ and $\theta = -1$, so

$$R(\mu) = \kappa(\mu)\left\{\pi^2 r(\mu)\kappa^*(\mu)|\psi_0(\mu)|^2 - \mathrm{i}\pi|\psi_0(\mu)|^2\lim_{\varepsilon\to+0}\int \frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}\varepsilon}\right.$$
$$+\mathrm{i}\pi\lim_{\varepsilon_1\to+0}\int \frac{\mathrm{d}x|\psi_0(x)|^2 r(x)\kappa^*(x)}{x-\mu-\mathrm{i}\varepsilon_1} + \lim_{\varepsilon\to+0}\lim_{\varepsilon_1\to+0}\int \frac{\mathrm{d}x|\psi_0(x)|^2}{x-\mu-\mathrm{i}\varepsilon_1}$$
$$\left.\times\int \frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{x-\nu+\mathrm{i}\varepsilon}\right\}.$$

In the double integral we change the order of integration and use the obvious relation $(x-\mu-\mathrm{i}\varepsilon_1)^{-1}(x-\nu+\mathrm{i}\varepsilon)^{-1}=[\mu-\nu+\mathrm{i}(\varepsilon+\varepsilon_1)]^{-1}[(x-\mu-\mathrm{i}\varepsilon_1)^{-1}-(x-\nu+\mathrm{i}\varepsilon)^{-1}]$. Then we get

$$R(\mu)=\kappa(\mu)\left\{\pi^2 r(\mu)\kappa^*(\mu)|\psi_0(\mu)|^2-\mathrm{i}\pi|\psi_0(\mu)|^2\lim_{\varepsilon\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}\varepsilon}\right.$$
$$\left.+\mathrm{i}\pi\lim_{\varepsilon_1\to+0}\int\frac{\mathrm{d}x|\psi_0(x)|^2 r(x)\kappa^*(x)}{x-\mu-\mathrm{i}\varepsilon_1}+I_1-I_2\right\}$$

$$I_1=\lim_{\varepsilon_1\to+0}\lim_{\varepsilon\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}(\varepsilon+\varepsilon_1)}\int\frac{\mathrm{d}x|\psi_0(x)|^2}{x-\mu-\mathrm{i}\varepsilon_1}$$

$$I_2=\lim_{\varepsilon_1\to+0}\lim_{\varepsilon\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}(\varepsilon+\varepsilon_1)}\int\frac{\mathrm{d}x|\psi_0(x)|^2}{x-\nu+\mathrm{i}\varepsilon}.$$

We choose in I_1 (I_2) the following order of passage to the limit: at first $\varepsilon_1\to+0$ ($\varepsilon\to+0$) and after this $\varepsilon\to+0$ ($\varepsilon_1\to+0$). Applying the Sohotsky formula for the internal integrals (in the inverse direction) we find

$$R(\mu)=\kappa(\mu)\left\{\pi^2 r(\mu)\kappa^*(\mu)|\psi_0(\mu)|^2-\mathrm{i}\pi|\psi_0(\mu)|^2\lim_{\varepsilon\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}\varepsilon}\right.$$
$$+\mathrm{i}\pi\lim_{\varepsilon_1\to+0}\int\frac{\mathrm{d}x|\psi_0(x)|^2 r(x)\kappa^*(x)}{x-\mu-\mathrm{i}\varepsilon_1}$$
$$+\lim_{\varepsilon\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}\varepsilon}\left[\mathrm{i}\pi|\psi_0(\mu)|^2+P\int\frac{|\psi_0(x)|^2\,\mathrm{d}x}{x-\mu}\right]$$
$$\left.-\lim_{\varepsilon_1\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}\varepsilon_1}\times\left[-\mathrm{i}\pi|\psi_0(\nu)|^2+P\int\frac{|\psi_0(x)|^2\,\mathrm{d}x}{x-\nu}\right]\right\}.$$

Collecting similar terms and using the relation

$$\kappa^*(\nu)P\int\frac{|\psi_0(x)|^2\,\mathrm{d}x}{x-\nu}=-\psi_0(\nu)\tag{60}$$

gives

$$R(\mu)=\kappa(\mu)\left\{\pi^2 r(\mu)\kappa^*(\mu)|\psi_0(\mu)|^2+P\int\frac{|\psi_0(x)|^2\,\mathrm{d}x}{x-\mu}\lim_{\varepsilon\to+0}\int\frac{r(\nu)\kappa^*(\nu)\,\mathrm{d}\nu}{\mu-\nu+\mathrm{i}\varepsilon}\right.$$
$$\left.+\lim_{\varepsilon_1\to+0}\int\frac{\mathrm{d}\nu\, r(\nu)\psi_0(\nu)}{\mu-\nu+\mathrm{i}\varepsilon_1}\right\}.$$

Applying once more the Sohotsky relation (in the 'inverse' direction) and using again relation (60) we arrive at the formula

$$R(\mu)=\kappa(\mu)\left\{\pi^2 r(\mu)\kappa^*(\mu)|\psi_0(\mu)|^2+P\int\frac{\mathrm{d}\nu\, r(\nu)\psi_0(\nu)}{\mu-\nu}\right.$$
$$\left.+P\int\frac{\mathrm{d}x|\psi_0(x)|^2}{x-\mu}P\int\frac{\mathrm{d}\nu\, r(\nu)\kappa^*(\nu)}{\mu-\nu}\right\}.$$

Substituting this relation into (59), we obtain the result

$$\int C(\mu,\nu)r(\nu)\,\mathrm{d}\nu=r(\mu)[1+\pi^2|\kappa(\mu)\psi_0(\mu)|^2].$$

Finally

$$C(\mu,\nu)=[1+\pi^2|\kappa(\mu)\psi_0(\mu)|^2]\delta(\mu-\nu).$$

Appendix B

Let us discuss the integral

$$G(s) = (2\pi \mathrm{i})^{-1} \int_L \mathrm{d}r \exp(rs) \left[r - G_0 - \int_\sigma \frac{p(\mu)\,\mathrm{d}\mu}{r-\mu} - \sum_{k=1}^{N} \frac{c_k^2}{r-\mu_k} \right]^{-1}. \tag{61}$$

Here σ is some interval of the real axis, $p(\mu)$ is a real valued smooth function $p(\mu) > 0$, c_k is a real constant, the contour of integration L is a straight line parallel to the imaginary axis and situated more right than σ(we assume at first that σ is bounded) and μ_k are real values. We are going to reduce this integral to an integral along the interval σ.

Equation (61) is a solution of the integral equation for the real values of s. However, we have to use it for the pure imaginary s also. In order to obtain this corresponding relation, we must rotate the contour of integration (taking into account the sign of s) and make it parallel to the real axis, $L \to L_1$. Then, we can deform the contour of integration into L_2 (see figure 3).

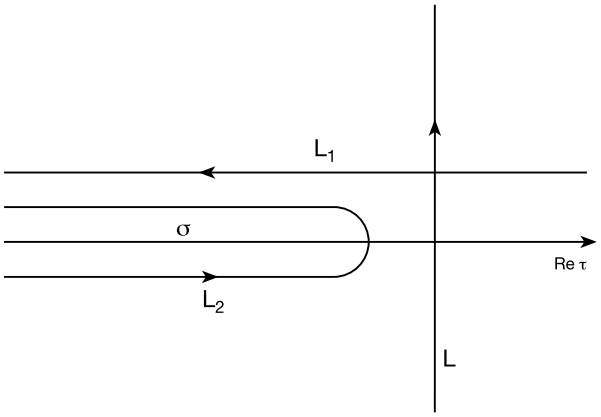

Figure 3.

Let us discuss the equation with respect to r

$$r = G_0 + P \int_\sigma \frac{p(\mu)\,\mathrm{d}\mu}{r-\mu} + \sum_{k=1}^{N} \frac{c_k^2}{r-\mu_k}. \tag{62}$$

This equation has some roots. If they are situated outside σ, then their contribution can be taken into account using the usual theory of residues. We are interested in the more complex situation, when these roots lie inside σ. Let $\zeta_m, m = 1, 2, \ldots, M$, be such roots. We assume for brevity, that they are simple roots of (62). Then the following limits exist

$$B_m = \lim_{r \to \zeta_m} \left\{ (r-\zeta_m) \left[r - G_0 - P \int \frac{\mathrm{d}\mu |\kappa(\mu) T(\mu)|^2}{r-\mu} - \sum_{k=1}^{N} \frac{|c_k|^2}{r-\mu_k} \right]^{-1} \right\}. \tag{63}$$

Then

$$G(s) = \sum_{m=1}^{M} \exp(s\zeta_m) B_m + \int_\sigma \mathrm{d}r \exp(sr) \left\{ \left[r - G_0 - P \int_\sigma \frac{p(\mu)\,\mathrm{d}\mu}{r-\mu} - \sum_{k=1}^{N} \frac{c_k^2}{r-\mu_k} \right]^2 + \pi^2 p^2(r) \right\}^{-1}. \tag{64}$$

At first we prove relation (64) in the case when $M = 1$, $\zeta_1 \in \sigma$. Let $V_\varepsilon = [\zeta_1 - \varepsilon, \zeta_1 + \varepsilon]$ and $\sigma_\varepsilon = \sigma \setminus V_\varepsilon$, $\varepsilon > 0$. Let us consider an equation

$$r = G_0 + P \int_{\sigma_\varepsilon} \frac{p(\mu)\,\mathrm{d}\mu}{r - \mu} + \sum_{k=1}^{N} \frac{c_k^2}{r - \mu_k} - 2\varepsilon p'(\zeta_1). \tag{65}$$

The last term here is the linear term with respect to ε of expansion of the integral

$$\int_{V_\varepsilon} \frac{p(\mu)\,\mathrm{d}\mu}{\zeta_1 - \mu}.$$

It follows from our construction, that equation (65) has a root $\zeta_1(\varepsilon) = \mathrm{Re}\,\zeta_1(\varepsilon)$, with $|\zeta_1(\varepsilon) - \zeta_1| = O(\varepsilon^2)$, when $\varepsilon \to 0$. Thus $\zeta_1(\varepsilon) \in V_\varepsilon$. Let us discuss an integral

$$G_\varepsilon(s) = (2\pi \mathrm{i})^{-1} \int_{L_2} \mathrm{d}r \exp(rs) \left[r - G_0 - \int_{\sigma_\varepsilon} \frac{p(\mu)\,\mathrm{d}\mu}{r - \mu} - \sum_{k=1}^{N} \frac{c_k^2}{r - \mu_k} + 2\varepsilon p'(\zeta_1) \right]^{-1}. \tag{66}$$

Because L_2 is separated from σ_ε, we get

$$G_\varepsilon(s) \to G(s) \qquad \text{when} \qquad \varepsilon \to 0. \tag{67}$$

Let us now calculate $G_\varepsilon(s)$ by deforming the contour. The integrand in $G_\varepsilon(s)$ has the cut on σ_ε and the pole $\zeta_1(\varepsilon)$. Using (63) we can introduce the value $B_1(\varepsilon)$. In addition $B_1(\varepsilon) \to B_1$ when $\varepsilon \to 0$. Representing the contour L_2 in (66) as a sum of contours enclosing cuts and pole separately (see figure 4) and evaluating the corresponding integrals in the usual way, we obtain

$$G_\varepsilon(s) = \exp[s\zeta_1(\varepsilon)] B_1(\varepsilon) + \pi \int_{\sigma_\varepsilon} \mathrm{d}r\, p^2(r) \exp(sr) \left\{ \left[r - G_0 - P \int_{\sigma_\varepsilon} \frac{p(\mu)\,\mathrm{d}\mu}{r - \mu} \right.\right.$$
$$\left.\left. - \sum_{k=1}^{N} \frac{c_k^2}{r - \mu_k} + 2\varepsilon p'(\zeta_1) \right]^2 + \pi^2 p^2(r) \right\}^{-1}.$$

In each term of the last expression we can pass to the limit for $\varepsilon \to 0$. Using equation (67) we find the final result

$$G(s) = \exp[s\zeta_1)] B_1 + \pi \int_{\sigma} \mathrm{d}r\, p^2(r) \exp(sr) \left\{ \left[r - G_0 - P \int_{\sigma} \frac{p(\mu)\,\mathrm{d}\mu}{r - \mu} \right.\right.$$
$$\left.\left. - \sum_{k=1}^{N} \frac{c_k^2}{r - \mu_k} \right]^2 + \pi^2 p^2(r) \right\}^{-1}. \tag{68}$$

The derivation for the case $M > 1$ is analogous.

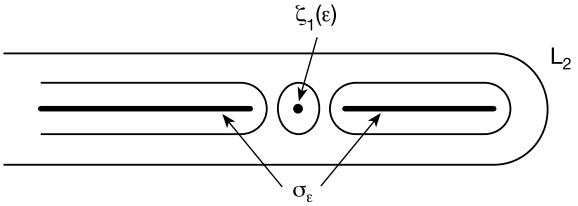

Figure 4.

We have obtained the last relation assuming that $p(\mu)$ has bounded support. However, if this function is fast decreasing when $|\mu| \to \infty$, then the denominator of the integrand on the right-hand side of (68) is decreasing like r^{-2} when $|r| \to \infty$. Since s is pure imaginary,

the integral in (68) is absolutely convergent. These considerations allow us the possibility to extend the validity of (68) and to use it for functions with infinite support, σ fast decreasing at infinity.

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